Taking the real distribution of V_t along the track into account improves the agreement between the theoretical and the experimental values of r_2.

The conformity of the graphs of selectivity and the value of r_{2exp} close to the calculated value unambiguously indicate that the suggested method of determining the size of micropores of nuclear filters is correct.

NOTATION

V_t, etching rate along the track; V_G, rate of etching through the inner pore surface; V_{G_1}, etching rate through the film surface; l, film thickness (length of track); r_1, r_2, r_{ef}, minimum, maximum, and effective pore radii, respectively; θ_0, half-angle of micropore taper; ρ, θ, running polar coordinates; z_i, coordinates of the micropore cross section; S_i, cross-sectional area; $\bar{P}_i$, pressure averaged over the micropore cross section; P, pressure difference on the membrane; q, power of the source; μ, dynamic viscosity; N, integral number of pores in the membrane; Q, volume of liquid flowing through the filter in unit time; R, selectivity; C_0, C, concentrations of the filtered substance in the initial preparation and in the filtrate, respectively; D, micropore diameter.

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INFLUENCE OF THE SHAPE OF A THIN INCLUSION ON THE TEMPERATURE DISTRIBUTION IN A PIECEWISE-HOMOGENEOUS PLANE

G. T. Sulim

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The plane stationary problem of heat conduction is considered without taking account of heat elimination through the side surfaces of a composite body with thin-walled interlayers on the interface of materials.

The structure of real materials is always without homogeneity and saturated by defects of the vacancy and impurity type, which often have the form of linear cracks or interlayers. Such inhomogeneities occur not only in the production stage of materials, but are also structural elements in the form of weld or glue connections. Hence, the development of simple and, if possible, exact methods of taking account of the influence of defects on the distribution of physicomechanical fields, particularly the temperature field, is of great value.

A system of N symmetric inclusions of the small thickness 2h are arranged on the abscissa axis $L = L' + L''$ of a Cartesian xOy coordinate system so that $L' = L_1 + \ldots + L_N$, where $L_n = [a_n, b_n]$ is the middle line of the n-th inclusion. The quantity $h = h(x)$ and $h(a_n) = h(b_n) = 0$ on the inclusion end faces. An ideal thermal contact with two half-planes S_2 and S_1 of different thermophysical properties, which are directly in contact on L'', is accomplished along the upper L_2' and lower L_1' boundaries of the interlayer. The thermal flux $q_1 + iq_2$ at infinity in the upper half-plane S_2, the thermal sources of intensity q_k^0 at the points $z_k = x_k + iy_k$ of the domains S_k, and the intensity q_0^0 at the point $z_0 = x_0$ on the axis of a

Ivan Franko Lvov State University.

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certain inclusion are all given. Find the temperature field of the host.

We represent the temperatures t and t_0 within the inclusion and the surrounding medium in the form

$$t(z) = \mathrm{Re}[m_0 \ln(z - z_0)] + t_*(z), \quad t_0(z) = t_1(z) + t_2(z), \tag{1}$$

where $t_1(z)$ is the main temperature field corresponding to the heat conduction problem for two half-planes in contact in the absence of inclusions and heat sources on the abscissa axis,

$$t_1(z) = -xq_{22} - yq_{1k} + m_k \ln|z - z_k| + m_k(n_k - n_l)\ln|z - \bar{z}_k| + 2k_l \ln|z - z_l| + t_1^0,$$

$$m_j = -\frac{q_j^0}{2\pi\lambda_j}, \quad n_j = \frac{\lambda_j}{\lambda_1 + \lambda_2}, \quad q_{ik} = \frac{q_i}{\lambda_k},$$
$$k_j = m_j n_j \, (z \in S_k; \; k, l = 1, 2; \; k \neq l; \; j = 0, 1, 2),$$

which satisfies the condition

$$t_1^+ = t_1^-, \quad \lambda_2 \frac{\partial t_1^+}{\partial y} = \lambda_1 \frac{\partial t_1^-}{\partial y} \quad (x \in L); \tag{2}$$

$t_*(z)$, $t_2(z)$ are the temperature perturbations within the inclusion and in the domain $S_1 \cup S_2$; t_1^0 is constant; λ_0, λ_1, λ_2 are heat conduction coefficients of the inclusions, the lower, and upper half-planes, respectively. Here and later, the superscripts + and − denote the limit values of the functions as the argument tends to the abscissa axis from the upper and lower half-planes, respectively, while the subscripts or the signs − and + also correspond to 1 and 2.

It is assumed that the inclusions do not affect the nature of the heat flux running parallel to their axis, and the thickness of the inclusions varies sufficiently smoothly so that the conditions for a perfect thermal contact between composite elements of the infinite plane can be represented as

$$t = t_0, \quad \lambda_k \frac{\partial t_0}{\partial y} = \lambda_0 \frac{\partial t}{\partial y} \quad (z \in L_k', \; k = 1, 2), \tag{3}$$

$$t_0^+ = t_0^-, \quad \lambda_2 \frac{\partial t_0^+}{\partial y} = \lambda_1 \frac{\partial t_0^-}{\partial y} \quad (z \in L''). \tag{4}$$

The accuracy of the second condition in (3) depends only on the angle of deviation of the normal to the line L_k' from the ordinate axis.

Using the smallness of the quantity h, the representation (1), the first condition in (3), and consequently the approximate equalities

$$\left.\frac{\partial t_*}{\partial y}\right|_{L_k'} = \frac{t_*(x + ih) - t_*(x - ih)}{2h}, \quad t_1(x \pm ih) = t_1^\pm \pm h \frac{\partial t_1^\pm}{\partial y}, \quad t_2(x \pm ih) = t_2^\pm,$$

we write the second condition in (3) in the form

$$\frac{\partial t_2^\pm}{\partial y} = -\beta \frac{\partial t_1^\pm}{\partial y} + \frac{\lambda_0}{2\lambda_k h}(t_2^+ - t_2^-) \pm \frac{p(x)}{2\lambda_k} \quad (k = 2, 1; \; x \in L'), \tag{5}$$

where

$$p(x) = \frac{2\lambda_0 m_0 h(x)}{(x - x_0)^2 + h^2(x)}; \quad \beta = 1 - \gamma; \quad \gamma = \frac{\lambda_0}{2n_1\lambda_2}$$

For convenience, the value of the perturbed field $t_2(z)$ is removed from the line L_k' to L' in writing the approximate equalities.

Taking account of (5), and also the second conditions of (4) and (2), we obtain

$$\lambda_2 \frac{\partial t_2^+}{\partial y} - \lambda_1 \frac{\partial t_2^-}{\partial y} = p(x) \quad (x \in L)$$

or

$$[\lambda_2 F^+(x) + \lambda_1 \overline{F^-(x)}] - [\lambda_1 F^-(x) + \lambda_2 \overline{F^+(x)}] = -2ip(x) \quad (x \in L). \tag{6}$$

Here the function F(z) satisfying the condition

$$t_2(z) = \mathrm{Re}(\int F(z)dz), \quad \left(\frac{\partial t_2(z)}{\partial x} = \mathrm{Re}F(z), \quad \frac{\partial t_2(z)}{\partial y} = -\mathrm{Im}F(z)\right) \tag{7}$$

has been introduced.

Let us examine the function of the jump in the perturbed temperature field $\gamma_2(x) = t_2^+ - t_2^-$. According to (2) and (4), it can be assumed that $\gamma_2(x) \equiv 0$, if $x \in L''$. Let the prime denote the derivative with respect to x; then $\gamma_2' \Delta x = t_2'^+ - t_2'^-$ or

$$[F^+(x) - \overline{F^-(x)}] - [F^-(x) - \overline{F^+(x)}] = 2\gamma_2'(x) \quad (x \in L). \tag{8}$$

The solution of linear conjugate problems of the boundary values (6) and (8) which damps out at infinity has the form ([1, Sec. 31])

$$F(z) = \frac{n_l}{\pi i} \int_{L'} \frac{\gamma_2'(t)\,dt}{t-z} + \frac{2k_0}{z - x_0 \pm ih}, \tag{9}$$

$$t_2(z) = \mathrm{Re}\left[\frac{n_l}{\pi i} \int_{L'} \frac{\gamma_2(t)dt}{t-z} + 2k_0 \ln(z - x_0 \pm ih)\right] \tag{10}$$

$$(z \in S_k; \ k, l = 1, 2; \ k \neq l).$$

The obvious condition

$$\gamma_2(a_n) = \gamma_2(b_n) = 0 \quad (n = 1, 2, \ldots, N) \tag{11}$$

was used in deriving (10).

According to the Sokhotskii–Plemelj formula (16.2) in [1], there frollows from (7), (9), and (10)

$$t_2^{\pm}(x) = \pm n_l\gamma_2(x) + 2k_0 \ln|x - x_0 \pm ih|, \quad \frac{\partial t_2^{\pm}}{\partial y} \frac{n_l}{\pi} \int_{L'} \frac{\gamma_2'(t)\,dt}{t-x} - \mathrm{Im}\,\frac{2k_0}{x - x_0 \pm ih},$$

and substituting these expressions into (5) yield a Prandtl integrodifferential equation,

$$\frac{1}{\pi} \int_{L'} \frac{\gamma_2'(t)\,dt}{t-x} - \alpha(x)\gamma_2(x) = g(x) \quad (x \in L'). \tag{12}$$

Here

$$g(x) = \beta g_1(x) + g_2(x), \quad \alpha(x) = \gamma/h(x).$$

$$g_1(x) = q_1 \frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2} + 2 \sum_{k=1,2} \frac{m_k y_k}{(x - x_k)^2 + y_k^2}, \quad g_2(x) = \frac{\lambda_1 - \lambda_2}{2\lambda_1\lambda_2} p(x).$$

The solution of (12) should satisfy the condition (11). Since $t_2(x \pm ih) = t_2^{\pm}(x)$ was assumed earlier, this approximation can then be weakened considerably if (10) is replaced by the relationship

$$t_2(z) = \mathrm{Re}\left[\frac{n_l}{\pi i} \int_{L'} \frac{\gamma_2(t)dt}{t - z \pm ih(t)} + 2k_0 \ln(z - x_0)\right] \quad (z \in S_k; \ k, l = 1, 2; \ k \neq l) \tag{13}$$

and (1) and (13) are later used in place of (10) in determining the function $t_0(z)$.

If the inclusions do not conduct heat ($\lambda_0 = 0$), then $\alpha(x) \equiv 0$, $g_2(x) \equiv 0$, and (12) is solved in closed form [1]:

$$\gamma_2'(x) = \frac{X^+(x)}{\pi} \int_{L'} \frac{\beta g_1(t)\,dt}{X^+(t)(t-x)} + X^+(x)P_{N-1}(x),$$

where

$$X(x) = \left[\prod_{n=1}^{N} (x - a_n)(x - b_n)\right]^{-\frac{1}{2}}, \quad P_{N-1}(x) = \sum_{n=1}^{N} c_n x^{n-1}.$$

The complex coefficients c_n are determined by using the condition (11). In the case of absolutely heat-conducting inclusions ($\lambda_0 = \infty$), there follows $\gamma_2(x) = h(x)g_1(x) \approx t_1(x - ih) - t_1(x + ih)$ from (12).

Having determined the jump $\gamma_1(x) = t_1(x + ih) - t_1(x - ih)$ in the fundamental temperature field $t_1(z)$ on the edges of the inclusions, we obtain that the jump $\gamma_1(x) + \gamma_2(x)$ in the temperature $t_0(z)$ on an absolutely heat-conducting inclusion is zero: the temperature within each such inclusion does not vary over the thickness.

For the thermophysical properties of the half-planes to be equal ($\lambda_1 = \lambda_2$), $g_2(x) \equiv 0$, the presence of heat sources on the axis of the inclusions in the homogeneous plane does not influence the temperature jump. Moreover, if the heat-conducting properties of the inclusions and host are identical, then $g(x) \equiv 0$, Eq. (12) determines $\gamma_2(x) \equiv 0$, and (1) and (13) yield the solution of the problem for a homogeneous plane,

$$t_0(z) = \sum_{j=0}^{2} m_j \ln|z-z_j| - xq_{22} - yq_{12} + t_1^0.$$

The previous approaches [2, 3] to the solution of problems about thin-walled inclusions—interlayers permitted the exact realization of just the passage to the limit for a thermally impermeable inclusion.

As an illustration, one inclusion located along the segment $[-a, a]$ of the real axis is investigated when $h(x) = h_0[1-(x/a)^2]^{1/2q}$ ($q \geqslant 1$). For $q = 1$ the interlayer is elliptical in shape, and rectangular for $q = \infty$. The solution of (12) for $L' = [-a, a]$ with the condition $\gamma_2(\pm a) = 0$ is sought in the form of the series

$$\gamma_2'(x) = \frac{a}{\sqrt{a^2-x^2}} \sum_{p=1}^{\infty} A_p T_p\left(\frac{x}{a}\right),$$
$$\gamma_2(x) = -\sqrt{a^2-x^2} \sum_{p=1}^{\infty} \frac{A_p}{p} U_{p-1}\left(\frac{x}{a}\right), \quad (|x| \leqslant a), \tag{14}$$

where $T_p(x)$, $U_{p-1}(x)$ are Chebyshev polynomials of the first and second kinds. As a result of applying the procedure of the method of orthogonal polynomials [4], we have an infinite system of linear algebraic equations to determine the unknown coefficients of the expansions (14):

$$\frac{\pi}{2} A_{k+1} + \alpha_0 \sum_{p=1}^{\infty} A_p H_{pk}^q = g_k \quad (k = 0, 1, \ldots), \tag{15}$$

where

$$H_{pk}^q = \frac{\pi\Gamma(2\nu)}{p4^\nu} \sum_{m=0,1} (-1)^m \frac{\sin(p_m \pi)}{\Gamma(1+\nu+p_m)\Gamma(\nu-p_m)}, \quad 2p_m = k + (-1)^m p,$$
$$g_k = \int_{-1}^{1} g(at)\sqrt{1-t^2}\, U_k(t)dt, \quad \alpha_0 = \frac{\gamma a}{h_0}, \quad \nu = 1 - \frac{1}{2q}.$$

Let us note that

$$H_{pk}^\infty = -\frac{4(k+1)\sin^2(p_0\pi)}{[(k+2)^2-p^2][k^2-p^2]}, \quad H_{pk}^1 = \frac{\pi}{2p}\delta_{p,k+1},$$
$$\delta_{p,k+1} = \begin{cases} 0, & (p \neq k+1); \\ 1, & (p = k+1). \end{cases}$$

It is easy to see that the system of equations (15) is quasi-completely regular for all physically possible values of the parameters and we apply the method of reduction to its solution [5]. For an inclusion of elliptical shape ($q = 1$), the solution of the system (15) is written explicitly: $A_p = 2pg_{p-1}/\pi(p+\alpha_0)$. In particular, if there are no heat sources, then $A_p = q_1\beta(\lambda_1+\lambda_2)\delta_{p-1}/\lambda_1\lambda_2(1+\alpha_0)$ and $\gamma_2(x) = -A_1\sqrt{a^2-x^2}$ ($|x| \leqslant a$).

Computations of the jump in the perturbed field $\gamma_2(x)$ on an inclusion in the homogeneous plane ($\lambda_1 = \lambda_2$) subjected to the effect of a homogeneous heat flux at infinity for $a/h_0 = 10$ and different values of the parameters q and $\lambda = \lambda_0/\lambda_1$ are performed on an ES-1022 electronic computer. In this case $A_{2p} = 0$ ($p = 1, 2, \ldots$) and it is sufficient to limit oneself to the first 20 nonzero coefficients in the expansions (14) to reach accuracy within 1% limits for $|x| \leqslant 0.95a$ in the most unfavorable case of a rectangular absolutely heat-conducting inclusion. The solutions $\gamma_2(x)$ obtained for $\lambda = 0.001$ and $\lambda = 1000$ differ from the corresponding analytic solutions for $\lambda = 0$ and $\lambda = \infty$ by less than 1%. Computations have shown that $\gamma_2(x) \geqslant 0$ for any values of the parameters ($\lambda_0 - \lambda_1$). If $\gamma_2^q(x)$ denotes the value of the function $\gamma_2(x)$ found for a definite value of q, then for real numbers $c \geqslant d \geqslant 1$ for each fixed λ the inequality $|\gamma_2^c(x)| \geq |\gamma_2^d(x)|$ holds, in particular, $|\gamma_2^\infty(x)| \geqslant |\gamma_2^q(x)| \geqslant |\gamma_2^1(x)|$.

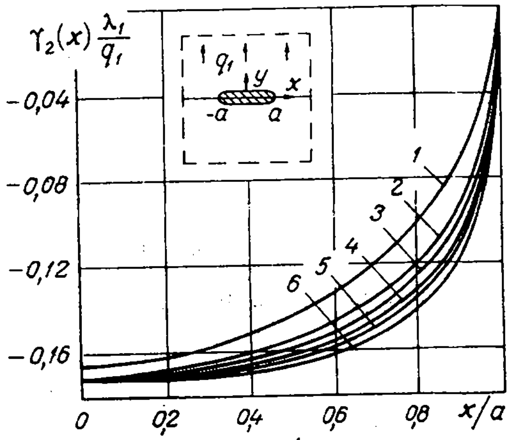

Fig. 1

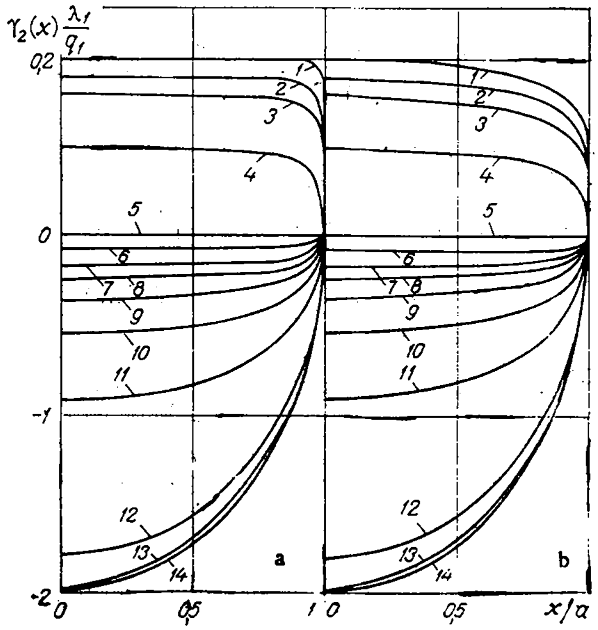

Fig. 2

Fig. 1. Change in the dimensionless quantity $\gamma_2(x)\lambda_1/q_1$ along the axis of an inclusion in a homogeneous plane for $\lambda_0/\lambda_1 = 0.5$ and some values of q: 1) q = 1; 2) 2; 3) 3; 4) 5; 5) 10; 6) 100.

Fig. 2. Change in the dimensionless quantity $\gamma_2(x)\lambda_1/q_1$ along the axis of an inclusion in a homogeneous plate: a) for q = 100; b) q = 4 [1) λ_0/λ_1 = 10,000; 2) 10; 3) 5; 4) 2; 5) 1; 6) 0.7; 7) 0.5; 8) 0.4; 9) 0.3; 10) 0.2; 11) 0.1; 12) 0.01; 13) 0.001; 14) 0.0001].

Varying the shape of the inclusion from the elliptical to the rectangular by using the parameter q changes $\gamma_2(0)$ by not more than 4%. This difference is maximal for $\lambda \simeq 0.1$ and tends to zero when $\lambda \to 0$ and $\lambda \to \infty$. The smaller the quantity λ, the smaller the influence of the shape of the inclusion on the temperature jump; for a heat-impermeable inclusion the temperature jump is independent of the shape. The dependence of $\gamma_2(x)$ on the quantity q is illustrated in Fig. 1 for $\lambda = 0.5$, and the dependence of $\gamma_2(x)$ on the parameter λ in Fig. 2.

NOTATION

L, real axis; $L' = L_1 + \ldots + L_N$, middle line of the inclusions; $L_n = [a_n, b_n]$, middle line of the n-th inclusion; L_2', L_1', upper and lower edges of the inclusions; S_2, S_1, upper and lower half-planes; h(x), half the inclusion thickness; h_0, half the inclusion thickness at its middle; z = x + iy, complex coordinate; $z_j = x_j + iy_j$, heat source coordinate; q_j^0, intensity of the j-th heat source; $q_1 + iq_2$, heat flux at infinity; λ_0, λ_1, λ_2, heat conduction coefficients of the inclusion, lower, and upper half-planes, respectively; t(z), $t_0(z)$, temperatures within and outside the inclusions; $t_*(z)$, $t_2(z)$, perturbed temperature fields inside and outside the inclusions; $t_1(z)$, t_1^0, main temperature field in the host in the absence of inclusions and its constant component; t_j^+, t_j^-, limit values of the function $t_j(z)$ (j = 0, 1, 2) on the abscissa axis for y > 0 and y < 0; $\gamma_k(x)$, jump in the function $t_k(z)$ on the inclusion; a, half the length of the inclusion in the example; q, a parameter characterizing the shape of the inclusion; $T_l(x)$, $U_l(x)$, Chebyshev polynomials of the first and second kinds; $\Gamma(x)$, gamma function; $\delta_{p,m}$, Kronecker delta; $\gamma_2^q(x)$, value of the function $\gamma_2(x)$, calculated for a definite q.

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